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THE MEAN RETURN TIME TO A STATE FOR A MARKOV CHAIN REPRESENTING N TELEPHONE CONVERSATIONS

By

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ABSTRACT

The mean return time to a state for a Markov chain representing N off-hook telephone conversations can be obtained by using ergodic flow rates [1] or renewal theory [2]. In this paper it is shown that the same result may be obtained by using no more knowledge of stochastic processes than the solution of time-homogeneous Chapman-Kolmogorov equations.

1.0 INTRODUCTION

One important quantity required in the modelling of telephone systems is the mean return time to a state for a Markov chain represetning N telephone conversations [2]. This result is derived in this paper by using no more knowledge of stochastic processes than the derivation and solution of Chapman-Kolmogorov equations.

2.0 MEAN RETURN TIME TO A STATE

We begin by assuming that each telephone call can be either in talkspurt or in silence and that the lengths of talkspurts and silences are both exponentially distributed with means β^{-1} and α^{-1} respectively. These assumptions mean that the aggregate process of N telephone calls can be represented by a Markov chain (Fig.1). A state of this Mrkov chain represents the number of calls in talksput. Our task in this paper is the calcuation of the mean return time to any state of this Markov chain.

Let t_k, k = 1,2, ... be the time points at which the k^{th} change of state of the Markov chain in Fig. 1 occurs and let $J(t_k)$ be the number of calls in talksput just after the k'th change of state. Let J(t) be a realization of the stochastic process whose Markov chain is shown in Fig. 1 and let (m,n) be an ordered pair of states from Fig.1 for which the transition rate from state m to state n is not zero. With the process J(t) we associate a process T(t) defined on a state space whose elements are all such ordered pairs such that for t_k, k = 1,2, ... the following events are equivalent

$$(J(t_k+)=n \,|\, J(t_k-)=m) \equiv \{T(t)=(m,n), t \in [t_k, t_{k-1})\} \qquad (1)$$

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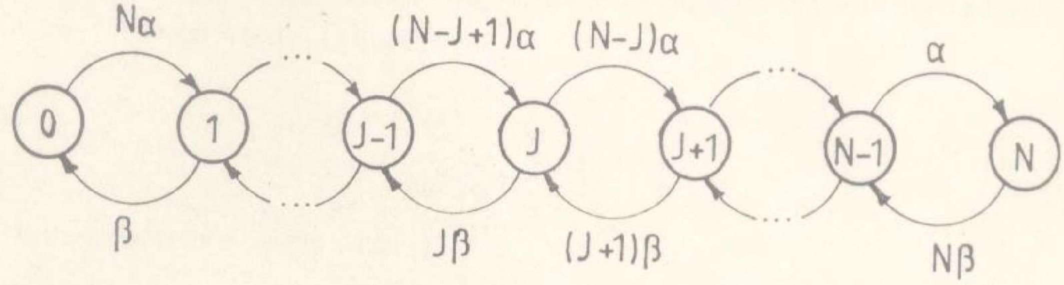

Note: Mean talkspurt length $= \beta^{-1}$

Mean silence length $= \alpha^{-1}$

Fig. 1. A Markov Chain Representing N Telephone Conversations.

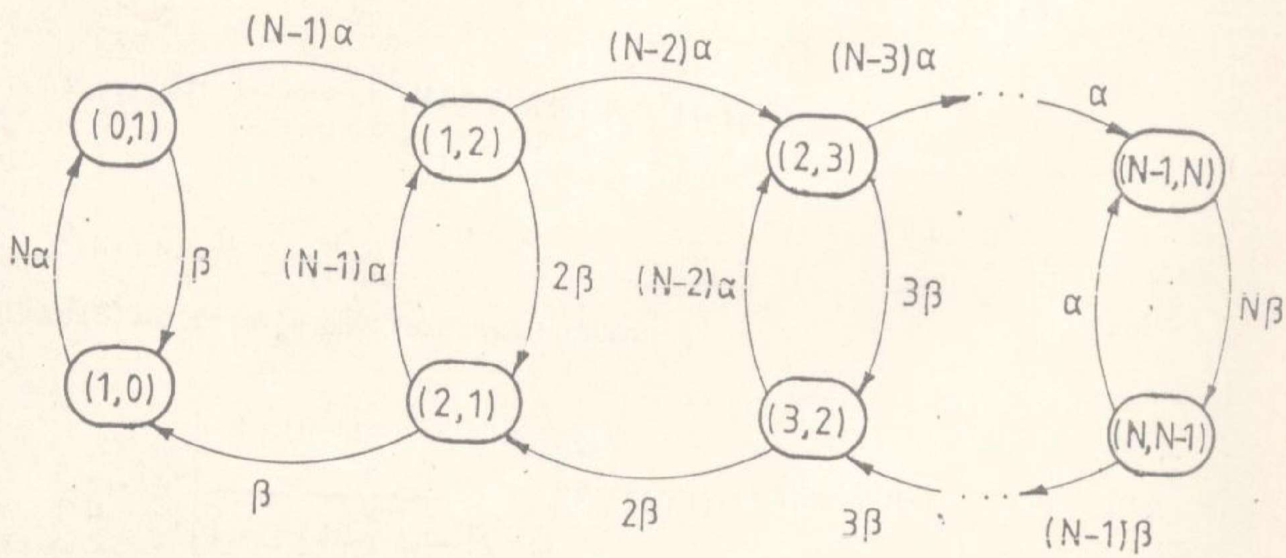

Fig. 2. A Markov Chain for the Process T (t).

We observe that $T(t)$ represents an ergodic Markov process whose transition diagram is shown in Fig. 2. The states of $T(t)$ in Fig. 2 are given by definition (1) above.

Let T_A be the mean dwell time of the prcess $T(t)$ in state (i,j) and let T_B be the mean time spent by the process $T(t)$ outside state (i,j) until return.

Then

$$\overline{T} = T_A + T_B \tag{2}$$

where $\overline{T}$ is the mean return time to state (i,j)

If we now let $P(i,j)$ be the ergodic probability of $T(t)$ being in state (i,j) then

$$P(i,j) = \frac{T_A}{T_A + T_B} = \frac{T_A}{\overline{T}} \tag{3}$$

which implies that

$$\overline{T} = \frac{T_A}{P(i,j)} \tag{4}$$

The time-homogeneous Chapman-Kolmogorov equations for $T(t)$ can be easily shown to be

$$P_{(i,i+1)}[(N-i-1)\alpha + (i+1)\beta] = (N-1)\alpha[P_{(i-1,i)} + P_{(i+1,i)}]$$
$$i = 0, 1, \ldots, N-1$$

with

$$P_{(-1,0)} = 0 \tag{5}$$

and

$$P_{(i,i-1)}[(N-i+1)\alpha + (i-1)\beta] = i\beta[P_{(i-1,i)} + P_{(i+1,i)}]$$
$$i = 1, 2, \ldots, N$$

with

$$P_{(N+1,N)} = 0 \tag{6}$$

Equations (5) and (6) can be solved recursively to obtain

$$P_{(i,i+1)} = \left[\frac{1 + (N-1)\frac{\alpha}{\beta}}{(i+1) + (N-i-1)\frac{\alpha}{\beta}}\right]\binom{N-1}{i}\left(\frac{\alpha}{\beta}\right)^i P_{(0,1)};\ i = 1, 2, \ldots, N-1 \tag{7}$$

$$P_{(i,i-1)} \left[\frac{1+(N-1)\frac{\alpha}{\beta}}{(i-1)+(N-i+1)\frac{\alpha}{\beta}}\right]\binom{N-1}{i-1}\left(\frac{\alpha}{\beta}\right)^{i-1} P_{(0,1)};\ i = 1,2,\ldots,\ N \qquad (8)$$

Using the normalizing condition that the sum of all probabilities must equal 1 we get

$$P_{(0,1)} = \frac{N\left(\frac{\alpha}{\beta}\right)}{\left[1+(N-1)\frac{\alpha}{\beta}\right]\left[1+\frac{\alpha}{\beta}\right]^{N}} \qquad (9)$$

We now note that in the state (i,j) we defined earlier either $j = i + 1$ or $j = i - 1$. Taking the case where $j = i + 1$ we note that the exponentially distributed dwell time in state $(i, i + 1)$ is

$$T_A = \frac{1}{(N-i-1)\alpha + (i-1)\beta} \qquad (10)$$

Hence

$$\bar{T} = \frac{T_A}{P_{(i,i+1)}} = \frac{\left[1+\frac{\alpha}{\beta}\right]^{N}}{\binom{N}{i}\left(\frac{\alpha}{\beta}\right)^{i}(N-i)\alpha}$$

But, we know that the ergodic probability, π_i, that i calls in Fig. 1 are talkspurt is given by

$$\pi_i = \frac{\binom{N}{i}\left(\frac{\alpha}{\beta}\right)}{\left[1+\frac{\alpha}{\beta}\right]^{N}} \qquad (12)$$

using eqn (12) in eqn (11) we get

$$\bar{T} = \frac{1}{\pi_i (N-i)\alpha}$$

which is the expression for the mean return time to state i for the Markov chain of Fig. 1

3.0 REFERENCES

1. J. Keilson, Markov Chain Models - Rarity and Exponentiality, Springer-Verlag, New York, 1979

2. M.L. Luhanga, "Analytical Modeling of a Packet Voice Concentrator", Ph.D. Thesis, Department of Electrical Engineering, Columbia University, New York, October 1984.